Integral Solutions - Integral Five
Expection of a Diffusion Process

Gary Schurman MBE, CFA

In this white paper we will build a diffusion model to estimate annualized cash flow at some future time t. Note that the diffusion mathematics that we will apply to annualized cash flow can also be applied to other diffusion processes like company value, stock price, interest-earning assets, etc. To this end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to estimate annualized cash flow given the following go-forward model assumptions...

Table 1: Model Parameters

Symbol	Description	Amount
C_0	Annualized cash flow at time zero ($)	1,000,000
μ	Cash flow growth rate - mean (%)	5.00
σ	Cash flow growth rate - volatility (%)	12.00

Our task is to answer the following question...

Question 1: What is expected annualized cash flow at the end of year five when the normally-distributed random variable has mean two and variance four (a non-standard diffusion process).

Question 2: What is expected annualized cash flow at the end of year five when the normally-distributed random variable has mean zero and variance one (a standard diffusion process).

The Equation for Random Annualized Cash Flow

We will define the variable C_t to be random annualized cash flow at time t and the variable δW_t to be the change in the driving brownian motion. Using the data in Table 1 above the equation for the change in random annualized cash flow over the infinitesimally small time interval $[t, t + \delta t]$ is... [2]

$$\delta C_t = \mu\, C_t\, t + \sigma\, C_t\, \delta W_t \text{ ...where... } \delta W_t \sim N\Big[0, \delta t\Big] \tag{1}$$

Equation (1) above states that random annualized cash flow over time follows a diffusion process. The solution to that stochastic differential equation (SDE) is the equation for random annualized cash flow at time t (unknown) as a function of annualized cash flow at time zero (known), which is... [2]

$$C_t = C_0 \operatorname{Exp}\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, z\right\} \text{ ...where... } z \sim N\Big[0, 1\Big] \tag{2}$$

The Equation for Expected Annualized Cash Flow

The equation for the normal curve where z is the random variable (independent variable) pulled from a normal distribution with mean m and variance v, and $f(z)$ is the height of the curve (dependent variable) at z, is... [1]

$$f(z) = \sqrt{\frac{1}{2\,\pi\, v}} \operatorname{Exp}\left\{-\frac{1}{2\,v}(z-m)^2\right\} \text{ ...where... } z \sim N\Big[m, v\Big] \text{ ...such that... } \operatorname{Prob}\Big[z\Big] = f(z)\,\delta z \tag{3}$$

Using Equation (3) above the equation for expected annualized cash flow at time t is...

$$\mathbb{E}\Bigg[C_t\Bigg] = \int\limits_{-\infty}^{\infty} f(z)\, C_t\, \delta z = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z-m)^2 \right\} C_t\, \delta z \tag{4}$$

Using Equation (2) above we can rewrite Equation (4) above as...

$$\mathbb{E}\Bigg[C_t\Bigg] = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z-m)^2 \right\} C_0\, \text{Exp}\left\{ \left(\mu - \frac{1}{2}\,\sigma^2\right) t + \sigma\,\sqrt{t}\,z \right\} \delta z \tag{5}$$

Using Appendix Equation (16) below the generalized equation for expected annualized cash flow at time t given that the normally-distributed random variable has mean m and variance v is...

$$\mathbb{E}\Bigg[C_t\Bigg] = C_0\, \text{Exp}\left\{ \mu\, t - \frac{1}{2}\,\sigma^2 t + m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t \right\} \tag{6}$$

Using Appendix Equation (17) below the generalized equation for expected annualized cash flow at time t given that the normally-distributed random variable has mean zero and variance one is...

$$\mathbb{E}\Bigg[C_t\Bigg] = C_0\, \text{Exp}\left\{ \mu\, t \right\} \text{ ...where... } m = 0 \text{ and } v = 1 \tag{7}$$

Answers To Our Hypothetical Problem

Question 1: What is expected annualized cash flow at the end of year five when the normally-distributed random variable has mean two and variance four (a non-standard diffusion process).

Using Equation (6) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\Bigg[C_5\Bigg] = 1{,}000{,}000 \times \text{Exp}\left\{ 0.05 \times 5 - \frac{1}{2} \times 0.12^2 \times 5 + 2 \times 0.12 \times \sqrt{5} + \frac{1}{2} \times 4 \times 0.12^2 \times 5 \right\} = 2{,}446{,}495 \tag{8}$$

Question 2: What is expected annualized cash flow at the end of year five when the normally-distributed random variable has mean zero and variance one (a standard diffusion process).

Using Equation (7) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\Bigg[C_5\Bigg] = 1{,}000{,}000 \times \text{Exp}\left\{ 0.05 \times 5 \right\} = 1{,}284{,}025 \tag{9}$$

Appendix

A. The solution to Equation (5) above is...

$$\begin{aligned}
\mathbb{E}\Bigg[C_t\Bigg] &= \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z-m)^2 \right\} C_0\, \text{Exp}\left\{ \left(\mu - \frac{1}{2}\,\sigma^2\right) t + \sigma\,\sqrt{t}\,z \right\} \delta z \\
&= C_0\, \text{Exp}\left\{ \left(\mu - \frac{1}{2}\,\sigma^2\right) t \right\} \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z-m)^2 \right\} \text{Exp}\left\{ \sigma\,\sqrt{t}\,z \right\} \delta z \\
&= C_0\, \text{Exp}\left\{ \left(\mu - \frac{1}{2}\,\sigma^2\right) t \right\} \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z^2 - 2\,m\,z + m^2 - 2\,v\,\sigma\,\sqrt{t}\,z) \right\} \delta z \\
&= C_0\, \text{Exp}\left\{ \left(\mu - \frac{1}{2}\,\sigma^2\right) t \right\} \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi\,v}}\, \text{Exp}\left\{ -\frac{1}{2\,v}\,(z^2 - 2\,(m + v\,\sigma\sqrt{t})\,z + m^2) \right\} \delta z
\end{aligned} \tag{10}$$

We will define the variable θ as follows...

$$\theta = z - (m + v\,\sigma\sqrt{t}) \;\text{ ...where... }\; \theta^2 = z^2 - 2\,(m + v\,\sigma\sqrt{t})\,z + m^2 + 2\,m\,v\,\sigma\sqrt{t} + v^2\sigma^2 t \tag{11}$$

Using the definition in Equation (11) above we can rewrite Equation (10) above as...

$$\begin{aligned}\mathbb{E}\Big[C_t\Big] &= C_0\,\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,(\theta^2 - 2\,m\,v\,\sigma\sqrt{t} - v^2\sigma^2 t\right\}\delta z\\ &= C_0\,\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\text{Exp}\left\{m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\}\delta z\\ &= C_0\,\text{Exp}\left\{\left(\mu - \frac{1}{2}\,\sigma^2\right)t\right\}\text{Exp}\left\{m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\delta z\\ &= C_0\,\text{Exp}\left\{\mu\,t - \frac{1}{2}\,\sigma^2 t + m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\delta z \end{aligned}\tag{12}$$

Using Equation (11) above we can make the following statement...

$$\frac{\delta\theta}{\delta z} = 1 \;\text{ ...such that... }\; \delta z = \delta\theta \tag{13}$$

Using Equatios (11) and (13) above we can rewrite Equation (12) above as...

$$\begin{aligned}\mathbb{E}\Big[C_t\Big] &= C_0\,\text{Exp}\left\{\mu\,t - \frac{1}{2}\,\sigma^2 t + m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\}\int\limits_{-\infty-m-v\,\sigma\sqrt{t}}^{\infty-m-v\,\sigma\sqrt{t}}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\delta\theta\\ &= C_0\,\text{Exp}\left\{\mu\,t - \frac{1}{2}\,\sigma^2 t + m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\}\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\delta\theta \end{aligned}\tag{14}$$

Note the solution to the following integral... [1]

$$\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi\,v}}\,\text{Exp}\left\{-\frac{1}{2\,v}\,\theta^2\right\}\delta\theta = 1 \tag{15}$$

Using Equation (15) above the solution to Equation (14) above as...

$$\mathbb{E}\Big[C_t\Big] = C_0\,\text{Exp}\left\{\mu\,t - \frac{1}{2}\,\sigma^2 t + m\,\sigma\sqrt{t} + \frac{1}{2}\,v\,\sigma^2 t\right\} \tag{16}$$

B. Using Appendix Equation (16) above the solution to Equation (5) above when the random variable z is normally-distributed with mean zero and variance one is...

$$\mathbb{E}\Big[C_t\Big] = C_0\,\text{Exp}\left\{\mu\,t - \frac{1}{2}\,\sigma^2 t + \frac{1}{2}\,\sigma^2 t\right\} = C_0\,\text{Exp}\Big\{\mu\,t\Big\} \;\text{ ...when... }\; m = 0 \text{ and } v = 1 \tag{17}$$

References

[1] Gary Schurman, *The Calculus of the Normal Distribution,* October, 2010.

[2] Gary Schurman, *Browian Motion - An Introduction to Stochastic Calculus,* February, 2012.